2023-24 MATH2048: Honours Linear Algebra II

Homework 1

Due: 2023-09-15 (Friday) 23:59

For the following homework questions, please give reasons in your solutions. Scan your solutions and submit it via the Blackboard system before due date.

1. (Friedberg 1.3 Q19) Let W_1 and W_2 be subspaces of a vector space V. Prove that $W_1 \cup W_2$ is a subspace of V if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

2. (Friedberg 1.3 Q21) Consider V as the set of sequences $\{a_n\}$ of real numbers. By Exercise 20 of Section 1.2, it is a vector space over $\mathbb{R}$ (No need to prove this). Show that the set of convergent sequences $\{a_n\}$ (i.e., those for which $\lim_{n\to\infty} a_n$ exists) is a subspace of the vector space V.

3. (Modification of Friedberg 1.3 Q24) Show that F^n is the direct sum of the subspaces

$$W_1 = \{(a_1, a_2, ..., a_n) \in F^n : a_1 + a_2 + ... + a_n = 0\}$$
$$W_2 = \{(a_1, a_2, ..., a_n) \in F^n : a_1 = a_2 = \cdots = a_{n-1} = 0\}.$$

4. (Friedberg 1.6 Q29a) Prove that if W_1 and W_2 are finite-dimensional subspaces of a vector space V, then the subspace W_1+W_2 is finite-dimensional, and $\dim(W_1+W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$. Hint: Start with a basis $\{u_1, u_2, ..., u_k\}$ for $W_1 \cap W_2$ and extend this set to a basis $\{u_1, u_2, ..., u_k, v_1, v_2, ...v_m\}$ for W_1 and to a basis $\{u_1, u_2, ..., u_k, w_1, w_2, ...w_p\}$ for W_2.

5. Let F_1 and F_2 be fields. A function $g \in \mathcal{F}(F_1, F_2)$ is called an even function if $g(-t) = g(t)$ for each $t \in F_1$ and is called an odd function if $g(-t) = -g(t)$ for each $t \in F_1$. By Q22 of Section 1.3 the set of all even functions $\mathcal{E}(F_1, F_2)$ in $\mathcal{F}(F_1, F_2)$ and the set of all odd functions $\mathcal{O}(F_1, F_2)$ in $\mathcal{F}(F_1, F_2)$ are subspaces of $\mathcal{F}(F_1, F_2)$ (no need to prove this).

Suppose $2 \neq 0$ in F_2. Show that $\mathcal{F}(F_1, F_2) = \mathcal{E}(F_1, F_2) \oplus \mathcal{O}(F_1, F_2)$. (When $2 = 0$ in F_2, what happens?)

The following are extra recommended exercises not included in homework.

1. Show that $M_{n\times n}(F)$ is a vector space when F is a field.

2. Let S be the set of all 2-tuple real numbers (x, y) such that $x \geq y$. Is S a vector space under the usual operations of vector addition and scalar multiplication?

3. Consider the vector space of all 2×2 matrices with real entries. Is the set of all 2×2 diagonal matrices a subspace of this vector space? Is the set of all 2×2 upper triangular matrices a subspace of this vector space?

4. Find a basis for the null space of the linear system represented by the following matrix: $\begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ 3 & 6 & -3 \end{bmatrix}$.

5. Find a basis for the null space of the linear system represented by the following matrix: $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$.

6. Are the vectors $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, and $\begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$ linearly independent?

   Are the vectors $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$ linearly independent?

7. Consider the following set of vectors in $\mathbb{C}^2$: $\begin{bmatrix} 1+i \\ i \end{bmatrix}$, $\begin{bmatrix} i \\ 1-i \end{bmatrix}$. Are these vectors linearly independent or dependent?

8. Determine if the set of functions $\{e^x, e^{2x}, e^{3x}\}$ is linearly independent on the interval $[0, \infty)$.